The University of Chicago

SOME SPECIAL CASES OF THE FLECNODE TRANSFORMATION OF RULED SURFACES

A DISSERTATION

SUBMITTED TO THE FACULTY
OF THE ODGEN GRADUATE SCHOOL OF SCIENCE
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

BY

JOHN WAYNE LASLEY, JR.

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INTRODUCTION

The general theory of non-developable ruled surfaces may be made to depend[1] upon a system

$$(1)\qquad \begin{cases} y''+p_{11}y'+p_{12}z'+q_{11}y+q_{12}z=0, \\ z''+p_{21}y'+p_{22}z'+q_{21}y+q_{22}z=0 \end{cases}$$

of two second order linear homogeneous differential equations, where the coefficients p_{ik}, q_{ik} are functions of x, and where the strokes indicate differentiation as to x. According to the fundamental theory of differential equations these equations have a general solution, a pair of functions y and z of x, which can be expressed as linear combinations with constant coefficients of the four particular solutions (y_i, z_i), $(i=1, 2, 3, 4)$ for which the determinant

$$(2)\qquad D=\begin{vmatrix} y_1' & y_2' & y_3' & y_4' \\ z_1' & z_2' & z_3' & z_4' \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}$$

does not equal to zero.

If we interpret y_i and z_i $(i=1, 2, 3, 4)$ as the homogeneous co-ordinates of two points in space, we obtain two curves

$$y_k=y_k(x), \qquad z_k=z_k(x) \qquad (k=1, 2, 3, 4).$$

The line joining those points which correspond to the same values of x generates a ruled surface. This ruled surface will not be a developable as a result of the hypothesis that D is not equal to zero.

Let us transform the system (1) by putting

$$(3)\qquad \eta=\alpha y+\beta z,\ \zeta=\gamma y+\delta z,\ \xi=\xi(x),$$

where α, β, γ, δ, ξ are arbitrary functions of x for which

$$\Delta=\alpha\delta-\beta\gamma\neq 0.$$

[1] E. J. Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces*, Leipzig, 1906, pp. 126 ff., hereafter referred to as W.

This transforms (1) into a new system of the same kind. The integrating ruled surfaces of the two systems are the same, but they are referred to a different pair of directrix curves and a different independent variable.

The fundamental invariants of (1) are[1]

$$
(4) \qquad \begin{cases} \theta_4 = I^2 - 4J, \quad \theta_{4\cdot 1} = 8\theta_4\theta_4'' - 9\theta_4'^2 + 8I\theta_4^2, \\ \theta_{10} = (I^2-4J)(K-I'^2) + (II'-2J')^2, \\ \theta_9 = \begin{vmatrix} u_{11}-u_{22} & u_{12} & u_{21} \\ v_{11}-v_{22} & v_{12} & v_{21} \\ w_{11}-w_{22} & w_{12} & w_{21} \end{vmatrix}, \end{cases}
$$

where

$$
(5) \qquad I = u_{11} + u_{22}, \quad J = u_{11}u_{22}, - u_{12}u_{21}, \quad K = v_{11}v_{22} - v_{12}v_{21},
$$

and

$$
(6) \qquad \begin{cases} u_{11} = 2p_{11}' - 4q_{11} + p_{11}^2 + p_{12}p_{21}, \\ u_{12} = 2p_{12}' - 4q_{12} + p_{12}(p_{11} + p_{22}), \\ u_{21} = 2p_{21}' - 4q_{21} + p_{21}(p_{11} + p_{22}), \\ u_{22} = 2p_{22}' - 4q_{22} + p_{22}^2 + p_{12}p_{21}, \end{cases}
$$

$$
(7) \qquad \begin{cases} v_{11} = 2u_{11}' + p_{12}u_{21} - p_{21}u_{12}, \\ v_{12} = 2u_{12}' + (p_{11} - p_{22})u_{12} - p_{12}(u_{11} - u_{22}), \\ v_{21} = 2u_{21}' - (p_{11} - p_{22})u_{21} + p_{21}(u_{11} - u_{22}), \\ v_{22} = 2u_{22}' - p_{12}u_{21} + p_{21}u_{12}, \end{cases}
$$

$$
(8) \qquad \begin{cases} w_{11} = 2v_{11}' + p_{12}v_{21} - p_{21}v_{12}, \\ w_{12} = 2v_{12}' + (p_{11} - p_{22})v_{12} - p_{12}(v_{11} - v_{22}), \\ w_{21} = 2v_{21}' - (p_{11} - p_{22})v_{21} + p_{21}(v_{11} - v_{22}), \\ w_{22} = 2v_{22}' - p_{12}v_{21} + p_{21}v_{12}. \end{cases}
$$

A tangent plane intersects a surface in a plane curve which has a double point at the point of contact. If one of the double-point tangents is an inflectional tangent to this plane curve, the point of contact is called a flecnode, a name due to Cayley.[2] From the point of view of Salmon,[3] followed here, the inflectional tangent at this point intersects four consecutive generators of our surface. On each generator there will be in general two flecnode points,

[1] W., pp. 102, 104, 112.

[2] A. Cayley, *Collected Mathematical Papers*, II, 29.

[3] *Cambridge and Dublin Mathematical Journal*, IV (1849), 252–60.

since four consecutive generators of a ruled surface have two straight-line intersectors. As x varies these two points trace a flecnode curve of two branches. The tangents to the asymptotic curves through the flecnode points generate a flecnode surface of two sheets. For any system of form (1) the flecnodes are obtained by factoring the quadratic covariant

$$C=u_{12}z^2-u_{21}y^2+(u_{11}-u_{22})yz.$$

When a ruled surface S is referred to its flecnode curves, $u_{12}=u_{21}=0$. We may without loss of generality take $p_{11}=p_{22}=0$. Under these conditions the system of differential equations for one sheet $F^{(1)}$ of the flecnode surface may be written[1]

$$(9)\quad \begin{cases} y''-2\dfrac{q_{12}}{p_{12}}y'-\rho'-q_{11}y+\dfrac{q_{12}}{p_{12}}\rho=0, \\ \rho''+[2(q_{11}+q_{22})-p_{12}p_{21}]y'-2\dfrac{q_{12}}{p_{12}}\rho'+\left[2q'_{11}-p_{12}q_{21}-4\dfrac{q_{12}}{p_{12}}q_{11}\right]y \\ \qquad\qquad -q_{22}\rho=0, \end{cases}$$

where

$$\rho=2y'+p_{12}z, \qquad \sigma=2z'+p_{21}y.$$

In precisely the same way the system for the second sheet $F^{(-1)}$ may be written, if in (9) we transpose the subscripts and replace y and ρ by z and σ, respectively. This system we shall denote by (10).

We shall call $F^{(1)}$ the first flecnode transform of S. We shall call $F^{(-1)}$ the minus first flecnode transform of S for the reason that[2] the first flecnode transform of $F^{(-1)}$ is S. The minus first transform of $F^{(1)}$ is S. The first transform of $F^{(1)}$ we shall call the second transform of S and denote it by $F^{(2)}$. Continuing in this way we obtain a suite of surfaces which we shall call the flecnode suite. Questions naturally arise as to the cases in which this suite either terminates or returns again into itself. This paper concerns itself with some of these questions.

The author wishes to express to Professor Wilczynski his deep appreciation for his genuine interest and helpful criticism.

[1] W., p. 153. [2] *Ibid.*, p. 178.

I. CASES OF TERMINATION

A flecnode tangent is not ordinarily tangent to the flecnode curve. If it were, since it is at the same time tangent to the asymptotic curve which passes through the flecnode point, the flecnode curve would be a straight line. The corresponding sheet of the flecnode surface then degenerates into a straight line. In this event any ruled surface made up of the lines intersecting this line may be called its flecnode surface. If we try to continue the flecnode suite, we are powerless to determine which of these surfaces to select. We shall say in this case that the flecnode suite terminates with its first transform, since it cannot be continued in unambiguous fashion. *A necessary and sufficient condition that the flecnode suite terminate with its first transform is* $\theta_{10}=0$, *i.e., the given ruled surface has a straight-line directrix.*

Since the ruled surface has a straight-line directrix let us take this line as one of the reference curves C_y. Let C_z be an arbitrary curve. We may write

$$\text{(11)} \qquad \begin{cases} y_1=1, & y_2=x, \quad y_3=0, \quad y_4=0, \\ z_1=f_1(x), & z_2=f_2(x), \quad z_3=f_3(x), \quad z_4=1. \end{cases}$$

Let us compute the system (1) for which these are the fundamental solutions. We obtain[1] the following system,

$$y''=0,$$

$$z''+\left(xf_1''-f_2''-x\frac{f_1'f_3''}{f_3'}+\frac{f_2'f_3''}{f_3'}\right)y'-\frac{f_3''}{f_3'}z'+\left(\frac{f_1'f_3''}{f_3'}-f_1''\right)y=0,$$

whence by (6)

$$u_{11}=0, \quad u_{12}=0, \quad u_{22}=-\tfrac{1}{2}\{f_3, x\},$$

$$u_{21}=2\{f_3, x\}(f_2'-xf_1')+2(x'+2)\left(f_1''-\frac{f_1'f_3''}{f_3'}\right)-3\frac{f_3''}{f_3'}(xf_1''-f_2'') \\ +2(xf_1'''-f_2'''),$$

[1] W., p. 128.

where $\{f_3, x\}$ indicates the Schwarzian derivative of f_3 with respect to x. The flecnodes are given by

$$\eta = y, \quad \zeta = -u_{21}y - u_{22}z. \tag{12}$$

The second flecnode point ζ is given by

$$\begin{cases} \zeta_1 = -u_{21} - u_{22}f_1\,, \\ \zeta_2 = -u_{21}x - u_{22}f_2\,, \\ \zeta_3 = -u_{22}f_3\,, \\ \zeta_4 = -u_{22}\,. \end{cases} \tag{13}$$

We obtain the following result: *The equations of the most general curve which can serve as the second branch of the flecnode curve of a ruled surface with a straight-line directrix can be obtained without any integration.*

If now we make upon y and z the transformation indicated by (12) we refer our surface to its flecnode curves. The resulting equivalent system of form (1) is

$$\begin{cases} \eta'' = 0, \\ \zeta'' + \left(u'_{21} - 2\dfrac{u_{21}u'_{21}}{u_{22}} + p_{22}u_{21} - p_{21}u_{22}\right)\eta' \\ \quad + \left(u''_{21} + \dfrac{u_{21}u''_{22}}{u_{22}} - 2u'_{21}\dfrac{u'_{22}}{u_{22}} - 2u'^2_{22}\dfrac{u_{21}}{u^2_{22}} + p_{22}u'_{21} - p_{22}\dfrac{u_{21}u'_{22}}{u_{22}}\right)\eta \\ \quad + \left(\dfrac{u''_{22}}{u_{22}} - p_{22}\dfrac{u'_{22}}{u_{22}} - 2\dfrac{u'^2_{22}}{u^2_{22}}\right)\zeta = 0. \end{cases} \tag{14}$$

Carpenter has shown[1] that when one branch of the flecnode curve is rectilinear the second branch of the flecnode curve may be plane. In this case he has shown that p_{21} can be determined except for two constants of integration. He has found, moreover, that the second branch of the flecnode curve may be a conic, in which special case the equations of form (1) take the form

$$\begin{cases} y'' = 0, \\ z'' + \frac{1}{4}cy' - \frac{1}{4}z = 0. \end{cases} \tag{15}$$

We have found that $\theta_{10} = 0$ is a condition for the termination of the flecnode suite with its first transform. Let us suppose $\theta_{10} \neq 0$

[1] A. F. Carpenter, "Ruled Surfaces Whose Flecnode Curves Have Plane Branches," *Transactions of the American Mathematical Society*, XVI (1915), 529, hereafter referred to as C.

and that the suite terminates with the second transform. We shall then have $\theta_{10}^{(1)}=0$. We desire to express this condition in terms of the invariants of the original form (1).

The differential equations for $F^{(1)}$ are given by (9), but in a form which is not convenient for the computation of its invariants. We proceed to transform (9) into an equivalent system for which $p_{11}^{(1)}=p_{22}^{(1)}=u_{12}^{(1)}=u_{21}^{(1)}=0$. To do this we first refer the surface to its flecnode curves. Furthermore, we shall specialize the independent variable so as to make $u=1$. To refer the surface to its flecnode curves we put

$$y=y, \quad Y=-\frac{p_{12}'}{p_{12}}y+\rho. \tag{16}$$

The resulting system is

$$\left\{\begin{aligned}
&y''-2\frac{p_{12}'}{p_{12}}y'-Y'-\left(\frac{p_{12}''}{p_{12}}-\frac{3}{2}\frac{p_{12}'^2}{p_{12}^2}+q_{11}\right)y+\frac{p_{12}'}{2p_{12}}Y=0,\\
&Y''+\left[2\frac{p_{12}''}{p_{12}}-\frac{p_{12}'^2}{p_{12}^2}+2(q_{11}+q_{22})-p_{12}p_{21}\right]y'\\
&\quad+\left[\frac{p_{12}'''}{p_{12}}-3\frac{p_{12}'p_{12}''}{p_{12}^2}+\frac{3}{2}\frac{p_{12}'^3}{p_{12}^3}-\frac{p_{12}'}{p_{12}}(q_{11}+q_{22})-\frac{p_{12}p_{21}'}{2}+2q_{11}'\right]y\\
&\quad-\left(\frac{1}{2}\frac{p_{12}'}{p_{12}^2}+q_{22}\right)Y=0.
\end{aligned}\right. \tag{17}$$

By the choice of suitable multipliers we may transform (17) into a system which preserves the condition $u_{12}^{(1)}=u_{21}^{(1)}=0$ and satisfies the further condition $p_{11}^{(1)}=p_{22}^{(1)}=0$. We accomplish this by putting

$$y=p_{12}\eta, \quad Y=Y, \tag{18}$$

where $p_{12}\neq 0$ on account of the assumption $\theta_{10}\neq 0$, a transformation which reduces (17) to

$$\left\{\begin{aligned}
&\eta''-\frac{1}{p_{12}}Y'-\left(\frac{1}{2}\frac{p_{12}'^2}{p_{12}^2}+q_{11}\right)\eta+\frac{p_{12}'}{2p_{12}^2}Y=0,\\
&Y''+\left[2p_{12}''-\frac{p_{12}'^2}{p_{12}}+2p_{12}(q_{11}+q_{22})-p_{12}^2p_{21}\right]\eta'\\
&\quad+\left[p_{12}'''-\frac{p_{12}'p_{12}''}{p_{12}}+\frac{1}{2}\frac{p_{12}'^3}{p_{12}^2}+p_{12}'(q_{11}+q_{22})-p_{12}p_{21}p_{12}'-\frac{1}{2}p_{12}^2p_{21}'+2q_{11}'p_{12}\right]\eta\\
&\quad-\left(\frac{p_{12}'^2}{2p_{12}^2}+q_{22}\right)Y=0.
\end{aligned}\right. \tag{19}$$

We are now in a position to compute conveniently the invariants of $F^{(1)}$. In particular we have[1]

$$\theta_{10}^{(1)} = u^{(1)4} p_{12}^{(1)} p_{21}^{(1)} ,$$

where we use the upper index 1 systematically for the quantities referring to the surface $F^{(1)}$. We find

$$\theta_{10}^{(1)} = -2\frac{p_{12}''}{p_{12}} - \frac{p_{12}'^2}{p_{12}^2} - 2(q_{11}+q_{22}) + p_{12}p_{21} . \tag{20}$$

Since $\theta_{10}^{(1)}$ is an invariant of the system (17) which is geometrically determined by (1), it must also be an invariant of (1). We have[2]

$$\left\{\begin{aligned} & p_{12}p_{21} = \theta_{10} , \quad \frac{p_{12}''}{p_{12}} = \frac{1}{16\theta_{10}^2}(8\theta_{10}\theta_{10}'' + 4\theta_9'\theta_{10} + \theta_9^2 - 4\theta_{10}'^2), \\ & q_{11}+q_{22} = \tfrac{1}{32}(16\theta_{10} - \theta_{4\cdot 1}), \quad \frac{p_{12}'}{p_{12}} = \frac{1}{4\theta_{10}}(2\theta_{10}' + \theta_9). \end{aligned}\right. \tag{21}$$

Substituting these values in (20) we obtain

$$\theta_{10}^{(1)} = \frac{1}{16\theta_{10}^2}(-16\theta_{10}\theta_{10}'' - 8\theta_9'\theta_{10} + 4\theta_9\theta_{10}' - \theta_9^2 + 12\theta_{10}'^2 + \theta_{10}^2\theta_{4\cdot 1}). \tag{22}$$

In order to put $\theta_{10}^{(1)}$ into a form in which its invariant character will be apparent we seek to replace the derivatives occurring in (22) by invariants and to put into evidence the isobaric property which has been masked by the assumption $\theta_4 = 1$. We find that

$$\theta_{10}^{(1)} = \frac{1}{16\theta_{10}^2}(3\theta_{15}^2 - 2\theta_{10}\theta_{20} - 2\theta_4^{\frac{1}{2}}\theta_9\theta_{15} + 2\theta_4^{\frac{3}{2}}\theta_{10}\theta_{14} - \theta_4^3\theta_9^2 + \theta_{10}^2\theta_{4\cdot 1}).$$

is an invariant which reduces to (22) under the assumption $\theta_4 = 1$. Consequently,

$$\theta_{30} \equiv 3\theta_{15}^2 - 2\theta_{10}\theta_{20} - 2\theta_4^{\frac{1}{2}}\theta_9\theta_{15} + 2\theta_4^{\frac{3}{2}}\theta_{10}\theta_{14} - \theta_4^3\theta_9^2 + \theta_{10}^2\theta_{4\cdot 1} = 0$$

is a necessary and sufficient condition for the flecnode suite to terminate with its second transform.

One naturally inquires about the case $\theta_{10}^{(-1)} = 0$. It is obtained from the foregoing case $\theta_{10}^{(1)} = 0$ by merely changing the sign of $\theta_4^{\frac{1}{2}}$.

[1] W., p. 119. [2] *Ibid.*, p. 120.

If $\theta_{30}=0$, we may use the methods of the first part of this section to compute the second branch of the flecnode curve on $F^{(1)}$. This curve is at the same time a branch of the flecnode curve on S. We can obtain both branches of the flecnode curve on S, by applying the minus first flecnode transformation to $F^{(1)}$. Thus we see that *the equations of both branches of the flecnode curve on S may be determined without any integration if the flecnode suite terminates with its second transform.*

II. CASES IN WHICH THE GENERATORS OF THE SECOND AND MINUS SECOND TRANSFORMS INTERSECT

Let us inquire whether the generators of the second and minus second flecnode transforms can intersect. Consider the quantities

$$
(23)\qquad \begin{cases} \rho^{(1)}=2y'-2\dfrac{q_{12}}{p_{12}}y-\rho\,, \\[2ex] \mu\ \ =2\rho'+[2(q_{11}+q_{22})-p_{12}p_{21}]y-2\dfrac{q_{12}}{p_{12}}\rho\,, \end{cases}
$$

formed from (9) just as ρ and σ were formed from (1). Since the original system has been taken in the form for which $p_{11}=p_{22}=u_{12}=u_{21}=0$, we have

$$
\begin{aligned} 2q_{11}&=p'_{12}\,, & 2p_{12}p_{21}-4(q_{11}+q_{22})&=u_{11}+u_{22}\,, \\ p_{12}z&=\rho-2y', & 2\rho'+p_{12}\sigma&=u_{11}y\,. \end{aligned}
$$

If we substitute these in (23) we find

$$
(24)\qquad \begin{cases} \rho^{(1)}=-\dfrac{p'_{12}}{p_{12}}y-p_{12}z\,, \\[2ex] \mu\ \ =\frac{1}{2}uy-\dfrac{p'_{12}}{p_{12}}\rho-p_{12}\sigma\,. \end{cases}
$$

Every point on the line $\rho^{(1)}\mu$ has the property[1] that the straight line joining it to the corresponding point on the generator of $F^{(1)}$ is an asymptotic tangent of $F^{(1)}$. This correspondence is expressed[2] by the fact that the corresponding variables undergo cogredient transformation. In particular, the line $\sigma^{(1)}Y$ will be the tangent to the asymptotic curve through the second flecnode point, and consequently a generator of $F^{(2)}$, if, and only if, $\sigma^{(1)}$ is that point on the line $\rho^{(1)}\mu$ corresponding to Y as a point on the line $y\rho$. But we have

$$
Y=u_{21}^{(1)}y-u^{(1)}\rho\,,
$$

[1] W., p. 146. [2] *Loc. cit.*

and therefore

$$\sigma^{(1)}=u_{21}^{(1)}\rho^{(1)}-u^{(1)}\mu\,.$$

Using (24) we find

$$(25)\qquad \sigma^{(1)}=\left(-2\frac{q_{12}}{p_{12}}u_{21}^{(1)}-\tfrac{1}{2}uu^{(1)}\right)y-p_{12}u_{21}^{(1)}z+2\frac{q_{12}}{p_{12}}u^{(1)}\rho+p_{12}u^{(1)}\sigma\,.$$

Any point on the line $\sigma^{(1)}Y$ is given by the expression

$$(26)\qquad \left[\left(-2\frac{q_{12}}{p_{12}}u_{21}^{(1)}-\tfrac{1}{2}uu^{(1)}\right)\alpha+u_{21}^{(1)}\beta\right]y-p_{12}u_{21}^{(1)}\alpha z+\left(2\frac{q_{12}}{p_{12}}u^{(1)}-u^{(1)}\beta\right)\rho$$
$$+p_{12}u^{(1)}\sigma\,.$$

Similarly any point on the line $\sigma^{(-1)}Z$ is given by

$$(27)\qquad \begin{cases} -p_{21}u_{21}^{(-1)}\gamma y+\left[\left(\tfrac{1}{2}uu^{(-1)}-2\frac{q_{21}}{p_{21}}u_{21}^{(-1)}\right)\gamma+u_{21}^{(-1)}\delta\right]z \\ +p_{21}u^{(-1)}\gamma\rho+\left(2\frac{q_{21}}{p_{21}}u^{(-1)}\gamma-u^{(-1)}\delta\right)\sigma\,, \end{cases}$$

where the upper index -1 refers to the system (10). The point P_Z is the second flecnode point on $F^{(-1)}$, and $\sigma^{(-1)}$ is the point on the line $\rho^{(-1)}\nu$ corresponding to it. The quantities $\rho^{(-1)}$ and ν are obtained from (10) just as ρ and σ are from (1).

In order that a point on the line given by (26) may also be on the line given by (27), we must have

$$(28)\qquad \begin{cases} \left(2\frac{q_{12}}{p_{12}}u_{21}^{(1)}+\tfrac{1}{2}uu^{(1)}\right)\alpha-u_{21}^{(1)}\beta-p_{21}u_{21}^{(-1)}\omega\gamma+{*}=0\,, \\ p_{12}u_{21}^{(1)}\alpha+{*}-\left(2\frac{q_{21}}{p_{21}}u_{21}^{(-1)}-\tfrac{1}{2}uu^{(-1)}\right)\omega\gamma+u_{21}^{(-1)}\omega\delta=0\,, \\ 2\frac{q_{12}}{p_{12}}u^{(1)}\alpha-u^{(1)}\beta-p_{21}u^{(-1)}\omega\gamma+{*}=0\,, \\ p_{12}u^{(1)}\alpha+{*}-2\frac{q_{21}}{p_{21}}u^{(-1)}\omega\gamma+u^{(-1)}\omega\delta=0\,, \end{cases}$$

where ω is a proportionality factor. This requires the vanishing of the determinant of the system, which after a combining of columns can be written in the form

$$(29)\qquad \begin{vmatrix} \dfrac{uu^{(1)}}{2} & -u_{21}^{(1)} & -p_{21}u_{21}^{(-1)} & 0 \\ p_{12}u_{21}^{(1)} & 0 & \dfrac{uu^{(-1)}}{2} & u_{21}^{(-1)} \\ 0 & u^{(1)} & p_{21}u^{(-1)} & 0 \\ p_{12}u^{(1)} & 0 & 0 & u^{(-1)} \end{vmatrix} = 0\,,$$

or

$$(30)\qquad 4p_{12}p_{21}(u_{21}^{(-1)}u^{(1)} - u_{21}^{(1)}u^{(-1)})^2 - (uu^{(-1)}u^{(1)})^2 = 0\,.$$

We proceed to express (30) in terms of the invariants of S. We have $u = u^{(-1)} = u^{(1)}$. Our condition (30) can be written

$$(31)\qquad 4p_{12}p_{21}(u_{21}^{(-1)} + u_{21}^{(1)})^2 - u^4 = 0\,.$$

Now we have[1]

$$u^2 = \theta_4\,,\quad 4q = -u\,,\quad \left(\frac{p'_{12}}{p_{12}} - \frac{p'_{21}}{p_{21}}\right) = \frac{\theta_4\theta_9^2}{4\theta_{10}^2}\,,$$

$$p_{12}p_{21} = \frac{\theta_{10}}{\theta_4^2}\,,\quad u_{21}^{(-1)} + u_{21}^{(1)} = 4q\left(\frac{p'_{12}}{p_{12}} - \frac{p'_{21}}{p_{21}}\right).$$

Substituting these values in (31) we find

$$(32)\qquad \vartheta_{18} \equiv \theta_9^2 - \theta_4^2\theta_{10} = 0\,.$$

In the foregoing we have assumed $\theta_4 \neq 0$, $\theta_{10} \neq 0$, cases discussed elsewhere in this paper.

If $\vartheta_{18} = 0$, we may recover again the determinant of the system (28) whose vanishing is a necessary and sufficient condition that the system have a non-trivial solution. The quantities α, β, $\omega\gamma$, $\omega\delta$ are, in fact, proportional to the cofactors of the elements of any row in the determinant of the coefficients. Since the cofactors of the corresponding elements in the rows are proportional we are led to a unique set of ratios unless all of these cofactors vanish, i.e., we have a definite point of intersection. Consequently, $\vartheta_{18} = 0$ *is a necessary and sufficient condition for the generators of the second and minus second transforms to intersect.*

[1] W., p. 119.

If we choose the independent variable so that $\theta_4=1$, the condition (30) becomes

$$(33) \qquad 4(p_{12}p'_{21}-p_{21}p'_{12})^2-p_{12}p_{21}=0.$$

We may solve this relation in symmetric fashion by putting

$$P=p_{12}p_{21}\,, \quad Q=\frac{p_{12}}{p_{21}}\,.$$

Then (33) gives upon integration

$$(34) \qquad Q=ce^{\int \frac{dx}{2\sqrt{P}}}\,.$$

If we assume $p_{11}=p_{22}=u_{12}=u_{21}=u-1=0$ (32) becomes

$$(35) \qquad \theta_9^2-\theta_{10}=0\,.$$

Under the same assumptions Carpenter[1] has shown that C_y is a conic if, and only if,

$$(36) \qquad \theta_9+2\theta'_{10}=0\,.$$

We shall assume $\theta_9\neq 0$, for if $\theta_9=0$ then by (35) $\theta_{10}=0$, a case discussed elsewhere in this paper. Differentiating (35) we have

$$(37) \qquad 2\theta_9\theta'_9-\theta'_{10}=0\,.$$

If now we eliminate θ'_{10} from (36) and (37) we find, since $\theta_9\neq 0$,

$$(38) \qquad 4\theta'_9+1=0\,.$$

Integrating (38) we get

$$\theta_9=-\tfrac{1}{4}(x+c)\,.$$

From (35) we find

$$\theta_{10}=\tfrac{1}{16}(x+c)^2\,.$$

Using the condition[2] that C_y be a plane curve we have

$$\theta_{4\cdot 1}=(x+c)^2+8\,.$$

Moreover, we have assumed $\theta_4=1$. Thus we have obtained expressions for the four fundamental invariants in terms of the

[1] C., p. 515. [2] *Ibid.*, p. 516.

independent variable and one arbitrary constant. We may now compute[1] explicitly the coefficients of (1). They lead us to the system

$$(39)\qquad \begin{cases} y''+dz'-\frac{1}{4}y=0\,, \\ z''+\frac{1}{16}(x+c)^2y'+\frac{1}{16d}(x-c)y=0\,, \end{cases}$$

where c and d are two arbitrary constants. However, one of these constants is not essential since

$$\bar{y}=ay\,,\quad \bar{z}=bz\,,\quad \xi=x+l\,,$$

the most general transformation which leaves our conditions undisturbed, serves when

$$a=\frac{1}{d}\,,\quad l=c\,,\quad b=\frac{1}{k}\,,$$

to remove d. The resulting system is

$$(40)\qquad \begin{cases} \bar{y}''+k\bar{z}'-\frac{1}{4}\bar{y}=0\,, \\ k\bar{z}''+\frac{\xi^2}{16}\bar{y}'+\frac{\xi}{16}y=0\,. \end{cases}$$

For every value of k, (40) defines a class of mutually projective ruled surfaces. We see then that *there exists a single infinity of classes of mutually projective ruled surfaces the generators of whose second and minus second flecnode transforms intersect, which have the additional property that the flecnode curve consists of two distinct branches, one of which is a conic.* It can readily be shown by a direct test that the system (40) has all of the properties attributed to it.

Let us inquire whether in addition to the foregoing the second branch of the flecnode curve may be plane. The necessary condition is[2]

$$(41)\qquad 2\theta_{19}\theta_{10}''-3\theta_{10}'^2+\theta_{10}^2=0\,.$$

But the values of θ_{10} determined above clearly do not satisfy (41). So we conclude that *there are no ruled surfaces whose second and minus second flecnode generators intersect, which have the property*

[1] W., p. 120. [2] C., p. 516.

that their flecnode curve consists of two distinct plane branches, one of which is a conic.

We proceed to compute the invariants of $F^{(1)}$ and $F^{(-1)}$ in terms of the invariants of S for the case $\vartheta_{18}=0$. For this purpose we shall use the form (19) in which the surface is referred to its flecnode curve and multipliers for the dependent variables have been chosen so as to remove certain derivatives. The equations for $F^{(-1)}$ may be obtained from (19) by transposing the subscripts provided the dependent variables ζ and Z are put in place of η and Y. We find the following values for the invariants of system (19)

$$
(42)\qquad \begin{cases}
\theta_4^{(1)}=1\,, \\
\theta_9^{(1)}=8\left[-\dfrac{p_{12}'''}{p_{12}}-2\dfrac{p_{12}'}{p_{12}}(q_{11}+q_{22})+\frac{3}{2}p_{21}p_{12}'+\frac{1}{2}p_{12}p_{21}'-2q_{11}'\right], \\
\theta_{10}^{(1)}=-2\dfrac{p_{12}''}{p_{12}}+\dfrac{p_{12}'^2}{p_{12}^2}-2(q_{11}+q_{22})+p_{12}p_{21}\,, \\
\theta_{4\cdot 1}^{(1)}=16\left(-2\dfrac{p_{12}''}{p_{12}}+3\dfrac{p_{12}'^2}{p_{12}^2}+p_{12}p_{21}\right).
\end{cases}
$$

The corresponding invariants for $F^{(-1)}$ may be obtained from these by transposing the subscripts. If $\vartheta_{18}=0$, we have

$$
(43)\qquad \begin{cases}
p_{12}p_{21}'=\frac{1}{4}(2\theta_{10}'+\theta_9)\,, \quad p_{21}p_{12}'=\frac{1}{4}(2\theta_{10}'-\theta_9)\,, \\
q_{11}+q_{22}=\frac{1}{32}(16\theta_{10}-\theta_{4\cdot 1})\,, \quad \dfrac{p_{12}''}{p_{12}}=\dfrac{1}{16\theta_{10}^2}(8\theta_{10}\theta_{10}''+2\theta_9\theta_{10}'+\theta_{10}-4\theta_{10}'^2)\,, \\
\dfrac{p_{12}'''}{p_{12}}=\dfrac{1}{64\theta_{10}^3}(32\theta_{10}^2\theta_{10}'''-48\theta_{10}\theta_{10}'\theta_{10}''-12\theta_9\theta_{10}'^2+16\theta_9\theta_{10}\theta_{10}''+24\theta_{10}'^3+\theta_9\theta_{10}),
\end{cases}
$$

giving the following values for $\theta_4^{(1)}$, $\theta_9^{(1)}$, etc., in terms of the invariants of S, upon the hypothesis that the independent variable is chosen so as to make $\theta_4=1$,

$$
(44)\qquad \begin{cases}
\theta_4^{(1)}=1\,, \\
\theta_9^{(1)}=\dfrac{1}{8\theta_{10}^3}(-32\theta_{10}^2\theta_{10}'''+48\theta_{10}\theta_{10}'\theta_{10}''+12\theta_9\theta_{10}'^2-16\theta_9\theta_{10}\theta_{10}''-24\theta_{10}'^3 \\
\qquad\qquad -\theta_9\theta_{10}+2\theta_{10}^2\theta_{10}'\theta_{4\cdot 1}+\theta_9\theta_{10}^2\theta_{4\cdot 1}+2\theta_{10}^3\theta_{4\cdot 1}')\,, \\
\theta_{10}^{(1)}=\dfrac{1}{16\theta_{10}^2}(-16\theta_{10}\theta_{10}''+12\theta_{10}'^2-\theta_{10}+\theta_{10}^2\theta_{4\cdot 1})\,, \\
\theta_{4\cdot 1}^{(1)}=\dfrac{1}{\theta_{10}^2}(-16\theta_{10}\theta_{10}''+8\theta_9\theta_{10}'+\theta_{10}+20\theta_{10}'^2+16\theta_{10}^3)\,.
\end{cases}
$$

For the computation of the invariants of $F^{(-1)}$ the equations (43) remain valid if we transpose the subscripts in the left members and replace θ_9 by $-\theta_9$ in the right members. Consequently, equations (44) with the latter change give us the corresponding invariants of $F^{(-1)}$.

If we abandon the hypothesis $\theta_4=1$, we find the following

$$(45)\quad \begin{cases} \theta_4^{(1)}=\theta_4\,, \\ \theta_9^{(1)}=\dfrac{1}{8\theta_4^{\frac{3}{2}}\theta_{10}^3}(4\theta_{10}^2\theta_{25}-3\theta_{10}\theta_{15}\theta_{20}+3\theta_4^{\frac{3}{2}}\theta_9\theta_{15}^2-2\theta_4^{\frac{3}{2}}\theta_9\theta_{10}\theta_{20}+3\theta_{15}^3 \\ \qquad\qquad -\theta_4^{\frac{13}{2}}\theta_9\theta_{10}-\theta_{10}^2\theta_{15}\theta_{4\cdot1}+\theta_4^{\frac{3}{2}}\theta_9\theta_{10}^2\theta_{4\cdot1}-\theta_{10}^3\hat{\theta}_{15})\,, \\ \theta_{10}^{(1)}=\dfrac{1}{16\theta_{10}^2}(-2\theta_{10}\theta_{20}+3\theta_{15}^2-\theta_4^5\theta_{10}+\theta_{10}^2\theta_{4\cdot1})\,, \\ \theta_{4.1}^{(1)}=\dfrac{1}{\theta_{10}^2}(-2\theta_{10}\theta_{20}-4\theta_4^{\frac{3}{2}}\theta_9\theta_{15}+\theta_4^5\theta_{10}+5\theta_{15}^2)\,, \end{cases}$$

where[1]

$$\theta_{15}=5\theta_{10}\theta_4'-2\theta_4\theta_{10}'\,,\qquad \hat{\theta}_{15}=5\theta_{4\cdot1}\theta_4'-2\theta_4\theta_{4.1}'\,,$$
$$\theta_{20}=15\theta_{10}\theta_4'-4\theta_4\theta_{15}'\,,\qquad \theta_{25}=5\theta_{20}\theta_4'-\theta_4\theta_{20}'\,.$$

To verify (45) it is sufficient to note that these formulae reduce to (44) when $\theta_4=1$, and that the right members of (45) are invariants of S.

The invariants of $F^{(-1)}$ are given by a system obtained from (45) by replacing θ_9 by $-\theta_9$.

Since $\theta_{10}^{(1)}=\theta_{10}^{(-1)}$ we conclude that *in case the generators of the second and minus second flecnode transforms intersect, the minus first flecnode surface belongs to a special linear complex if, and only if, the first flecnode surface belongs to a special linear complex. In this event the second and minus second flecnode surfaces are straight lines having a point in common.*

Let us inquire whether the corresponding absolute invariants of $F^{(1)}$ and $F^{(-1)}$ may not have the same values. Equations (44) and the corresponding ones for $F^{(-1)}$ show that it will then be necessary that

$$(46)\qquad \begin{cases} \theta_9\theta_{10}'=0\,, \\ 12\theta_9\theta_{10}'^2-16\theta_9\theta_{10}\theta_{10}''-\theta_9\theta_{10}+\theta_9\theta_{10}^2\theta_{4\cdot1}=0\,. \end{cases}$$

[1] W., p. 112.

We shall again assume that $\theta_9 \neq 0$. Since $\vartheta_{18}=0$ we must then have also $\theta_{10} \neq 0$. The equations (46) reduce to

$$\theta'_{10}=0\,, \quad \theta_{10}\theta_{4\cdot 1}=1\,. \tag{47}$$

Equations (44) show that in this case

$$\theta_9^{(1)}=\theta_{10}^{(1)}=\theta_9^{(-1)}=\theta_{10}^{(-1)}=0\,,$$

so we conclude that the *absolute invariants of the first and minus first flecnode transforms are equal if, and only if, these surfaces belong to special linear complexes.*

III. CASES OF PERIODICITY

It has been shown[1] that each sheet of the flecnode surface of S has S itself as one of the sheets of its flecnode surface. Thus the minus first transform of $F^{(1)}$ is S. The first transform will ordinarily be a new surface $F^{(2)}$. Let us inquire whether it too can coincide with S. If so, the flecnode suite will be periodic, of period two. In this event the two sheets of the flecnode surface of $F^{(1)}$ must coincide. Then the two sheets of the flecnode surface of S cannot be distinct, else[1] they would be distinct on $F^{(1)}$. Thus we see that $\theta_4=0$ is a necessary condition. It is evident geometrically that this condition is also sufficient. We conclude then that *the flecnode suite is periodic, of period two, when, and only when, the flecnode curve meets every generator in two coincident points, or is indeterminate.*

This theorem may also be established analytically. The ruled surfaces for which the flecnode curves are indeterminate are quadrics. In this case we have not only $\theta_4=0$, but $u_{12}=u_{21}=u_{11}-u_{22}=0$. The flecnode surface in this case coincides with the original quadric generated by the generators of the second kind. The second flecnode surface is the original surface generated again by its first set of rulings.

We may now assume $\theta_4\neq 0$ and $\theta_{10}\neq 0$ since the cases excluded by this hypothesis have been considered already. The generators of the first and minus first transforms are[2] generators of the second kind on the quadric which osculates our original ruled surface along a generator. They are distinct since we have assumed $\theta_4\neq 0$. Consequently, they cannot intersect and, therefore, *the flecnode suite cannot be periodic, of period three.*

This theorem, too, may be established analytically. We proceed to determine whether the flecnode suite can be of period four. We shall assume again $\theta_4\neq 0$ and $\theta_{10}\neq 0$. We require that the line joining the point Y to the point Z be a generator of $F^{(2)}$.

[1] W., p. 178. [2] *Ibid.*, p. 147.

In particular, it is necessary that the point $\sigma^{(1)}$, given by (25), shall be a point on this line. Now, any point on the line YZ is given by the expression

$$u_{21}^{(1)}\alpha y+u_{21}^{(-1)}\beta z-u^{(1)}\alpha\rho-u^{(-1)}\beta\sigma\,.$$

Consequently, we must have

$$(48)\qquad \begin{cases} u_{21}^{(1)}\alpha+\left(2\dfrac{q_{12}}{p_{12}}u_{21}^{(1)}+\frac{1}{2}uu^{(1)}\right)\omega=0\,, \\ u_{21}^{(-1)}\beta+p_{12}u_{21}^{(1)}\omega=0\,, \\ u^{(1)}\alpha+2\dfrac{q_{12}}{p_{12}}u^{(1)}\omega=0\,, \\ u^{(-1)}\beta+p_{12}u^{(1)}\omega=0\,. \end{cases}$$

Since the three quantities α, β, ω cannot all be zero, it is necessary that the second order determinants

$$(49)\qquad \begin{vmatrix} u_{21}^{(-1)} & p_{12}u_{21}^{(1)} \\ u^{(-1)} & p_{12}u^{(1)} \end{vmatrix}\,,\qquad \begin{vmatrix} u_{21}^{(1)} & 2\dfrac{q_{12}}{p_{12}}u_{21}^{(1)}+\frac{1}{2}uu^{(1)} \\ u^{(1)} & 2\dfrac{q_{12}}{p_{12}}u^{(1)} \end{vmatrix}$$

shall both vanish. Now $p_{12}\neq 0$, else the hypothesis $\theta_{10}\neq 0$ is contradicted. Moreover, $u^{(1)}\neq 0$, since the flecnode curves are distinct on S, and are therefore distinct also on $F^{(1)}$. It is necessary then that

$$(50)\qquad u_{21}^{(-1)}u^{(1)}-u_{21}^{(1)}u^{(-1)}=u=0\,.$$

However, $u=0$ implies $\theta_4=0$, contrary to hypothesis. So we must conclude that *the flecnode suite cannot be periodic, of period four.*

VITA

John Wayne Lasley, Jr., was born in Burlington, North Carolina, September 22, 1891. After attending the public schools of his native town he entered the University of North Carolina in 1906, graduating with the degree of Bachelor of Arts in 1910. He received the degree of Master of Arts from this institution in 1911. In 1915–16 he attended the Johns Hopkins University, studying with Professors Bateman, Coble, Cohen, and Morley. During the summers of 1917, 1918, 1919, and the year 1919–20, he was in residence at the University of Chicago, studying with Professors Birkhoff, Bliss, Dickson, Moore, and Wilczynski. Since receiving his Bachelor's degree during the years not specified above he has taught in the University of North Carolina in the capacity of instructor, assistant professor, and associate professor. In connection with this work he studied with Professors Cain and Henderson. To all of his teachers he is indebted for inspiration and instruction. To Professor Wilczynski he is particularly grateful for sympathetic interest and helpful advice during the preparation of this dissertation.

Lasley, J. W.
Some special cases of the flecnode transformation of rules surfaces